Examples of Limits and the E-method

In the following pages, we will study many examples of limits and the E-method. It is basically an English adaptation from the Chinese translation edition "A Course on Mathematical Analysis" (xxxxxx, Chinese pinyin: shu xue fen xi jiao cheng) of the Russian textbook by the Russian authors Γ peóeHyd (xxxxxx, Chinese pinyin: ge lie ben ka) and HoBoceJIoB (xxxxxx, Chinese pinyin: nuo wo she nuo fu).

We emphasize that only very basic concepts and results of elementary functions are used (indeed no knowledge of derivatives are used).

Examples of Limits of Sequences

1. **The Euler Constant.** Note that for each $n \in \mathbb{N}$, because

$$\ln(n+1) - \ln n = \int_n^{n+1} \frac{1}{x}\, dx,$$

and because

$$\frac{1}{n+1} \le \frac{1}{x} \le \frac{1}{n} \quad \text{for } x \in [\frac{1}{n+1}, \frac{1}{n}],$$

we have:

$$\ln(n+1) - \ln n < \frac{1}{n}\,, \tag{1}$$

$$\frac{1}{n+1} < \quad \ln(n+1) - \ln n\,. \tag{2}$$

Adding up the above Inequalities (1) for $n = 1, 2, 3, \cdots, k$, we obtain:

$$\ln(k+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{k}.$$

Therefore,

$$a_{k+1} \overset{\textbf{def}}{=} 1 + \frac{1}{2} + \cdots \frac{1}{k} + \frac{1}{k+1} - \ln(k+1) > 0.$$

Also by Inequality (2) for $n = k$,

$$a_k - a_{k+1} = -\frac{1}{k+1} + \ln(k+1) - \ln k > 0.$$

We conclude that the sequence $\{a_n\}_{n=1}^{\infty}$ is monotone decreasing and bounded below, so it converges to a finite limit:

$$C \overset{\textbf{def}}{=} \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots \frac{1}{n} - \ln(n)\right).$$

We call C the Euler constant, and $C = 0.5772\cdots$.

2. Find

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}\right).$$

Let C be the Euler constant. From the above, we have:

$$1 + \frac{1}{2} + \cdots \frac{1}{2n} = \ln(2n) + C + \alpha_1$$

where $\lim_{n\to\infty} \alpha_1 = 0$, also,

$$1 + \frac{1}{2} + \cdots \frac{1}{n-1} = \ln(n-1) + C + \alpha_2$$

where $\lim_{n\to\infty} \alpha_2 = 0$. Thus

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} = \ln 2n - \ln(n-1) + \alpha_2 - \alpha_1 = \ln\frac{2n}{n-1} + \alpha_1 - \alpha_2.$$

Let $n \to \infty$, we get

$$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}\right) = \ln 2.$$

3. Suppose $\lim_{n\to\infty} a_n = a$, show that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a.$$

Let $s_n = a_1 + a_2 + \cdots + a_n$. We observe the following fact: If every one of m numbers $k_1, k_2, \cdots k_m$ lies in (c, d), then the Arithmetic Mean formed by these m numbers will also lie in (c, d). Now, let N be a natural number. When $n > N$, we have

$$\frac{s_n}{n} = \frac{s_N + (s_n - s_N)}{n} = \frac{s_N}{n} + \frac{a_{N+1} + \cdots + a_n}{n-N}\left(1 - \frac{N}{n}\right).$$

We can choose a sufficiently large N such that $a_{N+1}, a_{N+2}, \cdots$ all lie in $(a - \varepsilon, a + \varepsilon)$. Thus the number $\dfrac{a_{N+1} + a_{N+2} + \cdots + a_n}{n-N}$ will also lie in $(a-\varepsilon, a+\varepsilon)$. Thus we can write $\dfrac{a_{N+1} + a_{N+2} + \cdots + a_n}{n-N}$ in the form : $a + \gamma$ where $|\gamma| < \varepsilon$. Now we have

$$\frac{s_n}{n} = \frac{s_N}{n} + (a+\gamma)\left(1 - \frac{N}{n}\right),$$

so

$$\left|\frac{s_n}{n} - a\right| \le \frac{|s_N|}{n} + |\gamma| + |a+\gamma|\frac{N}{n} \le \frac{|s_N|}{n} + |\gamma| + (|a| + \varepsilon)\frac{N}{n}.$$

After fixing N, we see that for sufficiently large n, the first and third terms will be less than ε. Hence, for sufficiently large n, we have

$$\left|\frac{s_n}{n} - a\right| < 3\varepsilon.$$

Therefore,

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a.$$

Now, consider the case $a = \infty$. Since for sufficiently large N, $a_{N+1}, a_{N+2}, \cdots$ are all greater than any given positive number M (no matter how large M is),

$$\frac{s_n}{n} > \frac{s_N}{n} + M\left(1 - \frac{N}{n}\right).$$

As

$$\lim_{n\to\infty} \frac{s_N}{n} = 0, \quad \lim_{n\to\infty}\left(1 - \frac{N}{n}\right) = 1,$$

we have

$$\lim_{n\to\infty} \frac{s_n}{n} = \infty,$$

as desired. The case for $a = -\infty$ is similar.

4. Let $a_1, a_2, \ldots$ be a positive sequence. Suppose $\lim_{n\to\infty} a_n = a$. We have

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = \lim_{n\to\infty} e^{\ln \sqrt[n]{a_1 a_2 \cdots a_n}}.$$

Since

$$\lim_{n\to\infty} \ln a_n = \begin{cases} \ln a & (\text{if } 0 < a < \infty) \\ -\infty & (\text{if } a = 0) \\ \infty & (\text{if } a = \infty), \end{cases}$$

therefore,

$$\lim_{n\to\infty} \ln \sqrt[n]{a_1 a_2 \cdots a_n} = \lim_{n\to\infty} \frac{\ln a_1 + \ln a_2 + \cdots + \ln a_n}{n} = \begin{cases} \ln a & (\text{if } \infty > a > 0) \\ -\infty & (\text{if } a = 0) \\ \infty & (\text{if } a = \infty). \end{cases}$$

In other words, we have proved

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = \begin{cases} a & (\text{if } 0 < a < \infty) \\ 0 & (\text{if } a = 0) \\ \infty & (\text{if } a = \infty). \end{cases}$$

5. Suppose $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = l$. Show that $\lim_{n\to\infty} \sqrt[n]{a_n} = l$ where $a_1, a_2, \ldots$ is a positive sequence.

In fact, this result follows from the above that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdots \frac{a_n}{a_{n-1}}} = l.$$

6. Calculate the limit:

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{n!}\ .$$

Let $a_n = \dfrac{n!}{n^n}$, then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1 + \dfrac{1}{n}\right)^n} = \frac{1}{e}.$$

It follows from the previous discussion that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{e}.$$

7. Let $x \in (0, \pi/2)$. Then

$$\lim_{n\to\infty} \left(\cos\frac{x}{2} \cos\frac{x}{2^2} \cdots \cos\frac{x}{2^n}\right) = \frac{\sin x}{x}\ .$$

Since

$$\cos\frac{x}{2} = \frac{\sin x}{2\sin\frac{x}{2}},\ \cos\frac{x}{2^2} = \frac{\sin\frac{x}{2}}{2\sin\frac{x}{2^2}},\ \cdots, \cos\frac{x}{2^n} = \frac{\sin\frac{x}{2^{n-1}}}{2\sin\frac{x}{2^n}}, \cdots,$$

$$\cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\frac{x}{2^n}}.$$

Since

$$\lim_{n\to\infty}\frac{\sin x}{2^n\sin\frac{x}{2^n}}=\lim_{n\to\infty}\frac{\sin x}{2^n E\dfrac{x}{2^n}}=\frac{\sin x}{x},$$

we have shown that

$$\lim_{n\to\infty}\left(\cos\frac{x}{2}\cos\frac{x}{2^2}\cdots\cos\frac{x}{2^n}\right)=\frac{\sin x}{x}\ .$$

8. Let $x\in[0,1]$. We will show that the sequence

$$y_1=x,\ y_2=x-\frac{y_1^2}{2},\ y_3=x-\frac{y_2^2}{2},\cdots,\ y_n=x-\frac{y_{n-1}^2}{2},\cdots$$

converges to $-1+\sqrt{2x+1}$.

Since $y_1,y_3\geq 0$ and since

$$y_1-y_2=\frac{y_1^2}{2},\quad y_2-y_3=\frac{y_2^2-y_1^2}{2},\quad y_3-y_4=\frac{y_3^2-y_2^2}{2},$$

adding up these equalities in pairs, we have

$$y_1-y_3=\frac{y_2^2}{2}\geq 0,\qquad y_2-y_4=\frac{y_3^2-y_1^2}{2}\leq 0.$$

Using mathematical induction, we can show that:

$$y_n\in[0,x],\ \ y_{2n+1}-y_{2n-1}\leq 0,\ \ y_{2n}-y_{2n+2}\leq 0,\ \ y_{2n}\leq y_{2n-1},$$

hence,

$$0\leq y_2\leq y_4\leq\cdots\leq y_{2n}\leq y_{2n+2}\leq\cdots\leq y_{2n+1}\leq y_{2n-1}\leq\cdots\leq y_5\leq y_3\leq y_1=x.$$

Thus the sequences:

$$y_1,y_3,\cdots\quad\text{and}\quad y_2,y_4,\cdots$$

are convergent. Let

$$\lim_{n\to\infty}y_{2n+1}=\ell,\qquad\lim_{n\to\infty}y_{2n}=\ell_1.$$

From the identity

$$y_n-y_{n+1}=\frac{y_n^2-y_{n-1}^2}{2}$$

with even n, we obtain by letting $n\to\infty$:

$$\ell_1-\ell=\frac{\ell_1^2-\ell^2}{2},$$

i.e.

$$(\ell_1-\ell)(\ell_1+\ell-2)=0.$$

Since $0\leq y_n\leq x$, $\ell<1$ and $\ell_1<1$ (to see this, note that if $x=1$, then $y_3=\frac{7}{8}$), hence $\ell+\ell_1-2<0$. Therefore, $\ell_1=\ell$, and the sequence $y_1,y_2,\cdots$ converges to ℓ. Moreover, from the equality: $y_n=x-\dfrac{y_{n-1}^2}{2}$, we have

$$\ell=x-\frac{\ell^2}{2},$$

hence

$$\ell = -1 + \sqrt{2x+1} \qquad (\text{ as } \ell \geq 0).$$

Remark: The proof perhaps becomes more transparent if we note that, with $y_0 = 0$, we have:

$$\begin{array}{rcl} 0 = y_0 & \leq & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \leq y_1 = x \leq 1, \\ y_0 \leq y_2 & = & \frac{1}{2}(y_0+y_1) + \frac{1}{2}(y_1-y_0)[1-\frac{1}{2}(y_0+y_1)] \leq y_1, \\ y_2 \leq y_3 & = & \frac{1}{2}(y_1+y_2) - \frac{1}{2}(y_1-y_2)[1-\frac{1}{2}(y_1+y_2)] \leq y_1, \\ & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_{2n-2} \leq y_{2n} & = & \frac{1}{2}(y_{2n-2}+y_{2n-1}) + \frac{1}{2}(y_{2n-1}-y_{2n-2})[1-\frac{1}{2}(y_{2n-2}+y_{2n-1})] \leq y_{2n-1}, \\ y_{2n} \leq y_{2n+1} & = & \frac{1}{2}(y_{2n-1}+y_{2n}) - \frac{1}{2}(y_{2n-1}-y_{2n})[1-\frac{1}{2}(y_{2n-1}+y_{2n})] \leq y_{2n-1}, \end{array}$$

for all $n \in \mathbb{N}$.

The E-method

Now we study the evaluation of limits by the *E-method.* We introduce the following notation:

Notation. *When a function satisfies* $\lim_{x\to a} f(x) = 1$, *we write: around* a,

$$f(x) = E.$$

To represent different functions that have 1 as their limit at the same point, we may use E, E_1, $E_2, \cdots$ in the above notation. We should point out that if $\lim_{x\to a} f(x) = l$ and $l \neq 0$, or $l \neq \pm\infty$, then

$$f(x) = lE,$$

because $\lim_{x\to a} \dfrac{f(x)}{l} = 1$ (so according to our convention: $\dfrac{f(x)}{l} = E$ around a).

Some Relations

We collect some important relations, which we will encounter frequently in the calculation of limits of elementary functions.

1. For polynomial:
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad (a_n \neq 0)$$
around the point ∞ (or $-\infty$), we have
$$P(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\cdot\frac{1}{x} + \frac{a_{n-2}}{a_n}\cdot\frac{1}{x^2}\cdots + \frac{a_0}{a_n}\cdot\frac{1}{x^n}\right).$$
Since the expression inside the bracket has limit 1 at $\pm\infty$, we can write: around $\pm\infty$,
$$P(x) = a_n x^n E.$$
Thus
$$\lim_{x\to\pm\infty} |P(x)| = \infty.$$

2. Suppose
$$R(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} \quad (a_n, b_m \neq 0).$$
From the above, we have around $\pm\infty$,
$$R(x) = \frac{a_n x^n E_1}{b_m x^m E_2} = \frac{a_n}{b_m} x^{n-m} E,$$
where $E = E_1/E_2$. Thus
$$\begin{aligned} \lim_{x\to\pm\infty} |R(x)| &= \infty, \quad n > m, \\ \lim_{x\to\pm\infty} R(x) &= 0, \quad n < m, \\ \lim_{x\to\pm\infty} R(x) &= \frac{a_n}{b_n}, \quad n = m. \end{aligned}$$

3. Since $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, we have: around the point 0,
$$\frac{\sin x}{x} = E, \quad i.e. \sin x = Ex.$$

4. As $1 - \cos x = 2\sin^2 \frac{x}{2}$, by the above we have that around 0
$$1 - \cos x = 2\left(\frac{x}{2}E_1\right)^2 = \frac{x^2}{2}E, \quad (E_1^2 = E)$$
or
$$\cos x = 1 - \frac{x^2}{2}E.$$

5. Around 0,
$$\tan x = Ex.$$

6. $$\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{y}{\sin y} = 1$$
where $y = \arcsin x$. Hence, we have around 0,
$$\arcsin x = Ex.$$

7. By the same reasoning, we have around 0,
$$\arctan x = Ex.$$

8. Note that $\lim_{x\to 0}(1+x)^{1/x} = e$, and
$$\frac{\ln(1+x)}{x} = \ln(1+x)^{\frac{1}{x}}.$$
By the continuity of the function ln,
$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = \ln e = 1.$$

Thus, around 0,

$$\ln(1+x) = Ex.$$

Similarly, since

$$\log_a(1+x) = \frac{\ln(1+x)}{\ln a},$$

we have

$$\log_a(1+x) = \frac{Ex}{\ln a}.$$

9. We study the behavior of e^x around 0. Let $e^x - 1 = z$. Then $\lim_{x\to 0} z = 0$, hence, $x = \ln(1+z) = zE_1$, or $z = x/E_1 = Ex$. Therefore, we have

$$e^x = 1 + Ex.$$

For the function a^x:

$$a^x = e^{x\ln a} = 1 + Ex\ln a.$$

The above two relations can be represented as follows:

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1, \qquad \lim_{x\to 0}\frac{a^x-1}{x} = \ln a.$$

10. We discuss the behavior of the function $(1+x)^\alpha$ around 0, where α is any real number. We have

$$(1+x)^\alpha = e^{\alpha\ln(1+x)} = e^{\alpha x E_1} = 1 + (\alpha x E_1)E_2,$$

so

$$(1+x)^\alpha = 1 + \alpha x E.$$

Examples:

$$\frac{1}{1+x} = 1 - xE, \quad \frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}xE, \quad \sqrt{1+x} = 1 + \frac{1}{2}xE, \quad \sqrt[n]{1+x} = 1 + \frac{1}{n}xE\ .$$

11. Now, we study the behavior of the function $F(x) = [1 + x\varphi(x)]^{\frac{\psi(x)}{x}}$ around 0 under the assumptions that $\lim_{x\to 0}\varphi(x) = k$ and $\lim_{x\to 0}\psi(x) = l$. We emphasize that no assumption is put on the differentibility of φ or ψ. Note that

$$F(x) = e^{\frac{\psi(x)}{x}\ln[1+x\varphi(x)]}.$$

From the assumptions:

$$\varphi(x) = E_1 k, \quad \psi(x) = E_2 l,$$

we have (c.f. discussion in 8),

$$\ln[1 + x\varphi(x)] = x\varphi(x)E_3 = kxE_1E_3,$$

or

$$F(x) = e^{klE_1E_2E_3} = e^{klE}.$$

Thus, we have proved that

$$\lim_{x\to 0} F(x) = e^{kl} \quad \text{or} \quad [1 + x\varphi(x)]^{\frac{\psi(x)}{x}} = e^{kl}E.$$

12. For any real $\beta > 0$, we have
$$\lim_{x\to\infty} \frac{e^x}{x^\beta} = \infty. \tag{3}$$
To see the above relation, we consider for sufficiently large x, the behavior of the function $\frac{e^x}{x^\beta}$. It suffices to prove (3) under the assumption that β is an integer as $\frac{e^x}{x^\beta} > \frac{e^x}{x^{[\beta]+1}}$.
Let $x's$ integral part $[x] = n > \beta + 1$. We observe the following inequalities:
$$\begin{aligned}\frac{e^x}{x^\beta} &> \frac{2^x}{x^\beta} > \frac{2^n}{(n+1)^\beta} \\ &= \frac{1+\binom{n}{1}+\binom{n}{2}+\cdots\binom{n}{\beta}+\binom{n}{\beta+1}\cdots\binom{n}{n}}{(n+1)^\beta} > \frac{\binom{n}{\beta+1}}{(n+1)^\beta} = \frac{n(n-1)\cdots(n-\beta)}{(\beta+1)!(n+1)^\beta}.\end{aligned}$$
As
$$\lim_{n\to\infty} \frac{n(n-1)\cdots(n-\beta)}{(\beta+1)!(n+1)^\beta} = \frac{1}{(\beta+1)!}\lim_{n\to\infty} n\left(1-\frac{2}{n+1}\right)\left(1-\frac{3}{n+1}\right)\cdots\left(1-\frac{\beta+1}{n+1}\right) = \infty,$$
we see that (3) is true.
Moreover, we can show that for any $\alpha, \beta > 0$
$$\lim_{x\to\infty} \frac{e^{\alpha x}}{x^\beta} = \infty.$$
In fact, putting $\alpha x = y$, we get
$$\lim_{x\to\infty} \frac{e^{\alpha x}}{x^\beta} = \alpha^\beta \lim_{y\to\infty} \frac{e^y}{y^\beta} = \infty.$$
Also, for $a > 1$
$$\lim_{x\to\infty} \frac{a^{\alpha x}}{x^\beta} = \infty, \qquad \lim_{x\to\infty} \frac{x^\beta}{a^{\alpha x}} = 0,$$
because
$$a^{\alpha x} = e^{\alpha x \ln a}.$$
Examples:
$$\lim_{x\to\infty} \frac{e^x}{x} = \infty, \qquad \lim_{x\to\infty} \frac{e^{\frac{x}{2}}}{x^2} = \infty.$$

13. For any real $\alpha, \beta > 0$, to find the limit
$$\lim_{x\to\infty} \frac{\ln^\alpha x}{x^\beta},$$
we let $\ln x = y$ and get
$$\lim_{x\to\infty} \frac{\ln^\alpha x}{x^\beta} = \lim_{y\to\infty} \frac{y^\alpha}{e^{\beta y}} = 0.$$
Putting $y = \frac{1}{x}$, we have from the above that
$$\lim_{x\to 0^+} x^\beta \ln^\alpha x = \lim_{y\to\infty} \frac{(-\ln y)^\alpha}{y^\beta} = 0.$$

We can also find $\lim_{x\to 0^+} \dfrac{e^{-\frac{\alpha}{x}}}{x^\beta}$ with the same assumptions on α and β as before. In fact, letting $x = \dfrac{1}{y}$, we have

$$\lim_{x\to 0^+} \frac{e^{-\frac{\alpha}{x}}}{x^\beta} = \lim_{y\to\infty} e^{-\alpha y} y^\beta = 0.$$

By the same method, we show that for positive β,

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}}}{x^\beta} = 0.$$

Examples

(i) Let $n \geq k$ be integers, find

$$\lim_{x\to 1} \frac{(x^n-1)(x^{n-1}-1)\cdots(x^{n-k+1}-1)}{(x-1)(x^2-1)\cdots(x^k-1)}.$$

By the substitution $x = 1+y$, $x^s - 1 = syE_s = s(x-1)E_s$ (c.f. discussion in 10 above), the above required limit equals to:

$$\lim_{x\to 1} \frac{n(n-1)\cdots(n-k+1)(x-1)^k E_n E_{n-1}\cdots E_{n-k+1}}{1\cdot 2\cdots k(x-1)^k E_1 E_2\cdots E_k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k} = \binom{n}{k}.$$

(ii)

$$\begin{aligned} \lim_{x\to 0} \frac{\sqrt[5]{1+3x^4} - \sqrt{1-2x}}{x+x^2} &= \lim_{x\to 0} \frac{1+\frac{3}{5}x^4E_1 - (1-xE_2)}{x+x^2} \\ &= \lim_{x\to 0} \frac{xE_2 + \frac{3}{5}x^4 E_1}{x+x^2} = 1. \end{aligned}$$

(iii)

$$\begin{aligned} &\lim_{x\to\infty} \left(\sqrt[k]{(x+a_1)(x+a_2)\cdots(x+a_k)} - x \right) \\ = &\lim_{x\to\infty} \left(\sqrt[k]{x^k + (\sum a_i)x^{k-1} + (\sum a_i a_j)x^{k-2} + \cdots + a_1a_2\cdots a_k} - x \right) \\ = &\lim_{x\to\infty} x \left(\sqrt[k]{1 + \frac{(\sum a_i)x^{k-1} + (\sum a_i a_j)x^{k-2} + \cdots + a_1a_2\cdots a_k}{x^k}} - 1 \right) \\ = &\lim_{x\to\infty} x \left(\frac{(\sum a_i)x^{k-1} + (\sum a_i a_j)x^{k-2} + \cdots + a_1a_2\cdots a_k}{kx^k} E \right) \\ = &\frac{\sum a_i}{k} = \frac{a_1+a_2+\cdots+a_k}{k}, \end{aligned}$$

by the discussion in 10.

(iv)

$$\lim_{x\to\infty} \left(\frac{3x^3+2}{3x^3+1} \right)^{x^3} = \lim_{x\to\infty} \left(1 + \frac{1}{3x^3+1} \right)^{x^3} = \lim_{x\to\infty} \left(1 + \frac{1}{3x^3} E \right)^{x^3} = e^{\frac{1}{3}},$$

by the discussion in 11.

(v)

$$\begin{aligned}\lim_{y\to\infty}\left(\cos\frac{x}{y}+\lambda\sin\frac{x}{y}\right)^y &= \lim_{y\to\infty}\left[\left(1-\frac{1}{2}\cdot\frac{x^2}{y^2}E_1\right)+\lambda\frac{x}{y}E_2\right]^y\\ &= \lim_{y\to\infty}\left[1+\lambda\frac{x}{y}E_2-\frac{1}{2}\cdot\frac{x^2}{y^2}E_1\right]^y\\ &= \lim_{y\to\infty}\left[1+\lambda\frac{x}{y}E\right]^y = e^{\lambda x},\end{aligned}$$

where we have applied the results of 3, 4 and 11.

(vi) Let $a_1, a_2, \cdots a_k > 0$. Then

$$\begin{aligned}&\lim_{y\to\infty}\left(\frac{(a_1)^{\frac{1}{y}}+(a_2)^{\frac{1}{y}}+\cdots(a_k)^{\frac{1}{y}}}{k}\right)^y\\ =\ &\lim_{y\to\infty}\left(\frac{k+\frac{1}{y}(\ln a_1E_1+\ln a_2E_2+\cdots+\ln a_kE_k)}{k}\right)^y\\ =\ &\lim_{y\to\infty}\left(1+\frac{1}{y}\cdot\frac{\ln a_1+\ln a_2+\cdots+\ln a_k}{k}E\right)^y\\ =\ & e^{\frac{\ln(a_1a_2\cdots a_k)}{k}} = \sqrt[k]{a_1a_2\cdots a_k}\ ,\end{aligned}$$

where results from 9 and 11 are used.

(vii) Let $a > 0$. To find

$$\lim_{y\to\infty} y^2\left(a^{\frac{1}{y}}-a^{\frac{1}{y+1}}\right),$$

we cannot simply use results in 10 to write:

$$\lim_{y\to\infty} y^2\left(a^{\frac{1}{y}}-a^{\frac{1}{y+1}}\right) = y^2\left(\frac{1}{y}\ln aE_1-\frac{1}{y+1}\ln aE_2\right),$$

because we would still have to determine whether the limit of the last expression exits or not. Rather, we consider first the following transformation:

$$a^{\frac{1}{y}}-a^{\frac{1}{y+1}} = a^{\frac{1}{y+1}}\left(a^{\frac{1}{y}-\frac{1}{y+1}}-1\right) = a^{\frac{1}{y+1}}\left(a^{\frac{1}{y(y+1)}}-1\right) = a^{\frac{1}{y+1}}\frac{\ln a}{y(y+1)}E.$$

Then we conclude that

$$\lim_{y\to\infty} y^2\left(a^{\frac{1}{y}}-a^{\frac{1}{y+1}}\right) = \lim_{y\to\infty}\left[a^{\frac{1}{y+1}}\ln a\cdot\frac{y^2}{y(y+1)}E\right] = \ln a.$$